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Scaling laws for fully developed turbulent shear flows. Part 1. Basic hypotheses and analysis

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The present work consists of two parts. Here in Part 1, a scaling law (incomplete similarity with respect to local Reynolds number based on distance from the wall) is proposed for the mean velocity distribution in developed turbulent shear flow. The proposed scaling law involves a special dependence of the power exponent and multiplicative factor on the flow Reynolds number. It emerges that the universal logarithmic law is closely related to the envelope of a family of power-type curves, each corresponding to a fixed Reynolds number. A skin-friction law, corresponding to the proposed scaling law for the mean velocity distribution, is derived.

In Part 2 (Barenblatt & Prostokishin 1993), both the scaling law for the velocity distribution and the corresponding friction law are compared with experimental data.

1. Introduction

Turbulent shear flows, statistically stationary, and homogeneous in the direction of the mean flow, are of basic interest for modern turbulence studies, both from a purely theoretical viewpoint and in connection with technical and geophysical applications. We shall speak specifically about shear flows bounded by a rigid wall, e.g. flows in tubes and channels, and more specifically about shear flows in circular cylindrical tubes, for which the best experimental data are available. Considering these rather special flows can help to shed light on wider classes of shear flows, such as boundary layers.

From the early 1930's on, two different forms were used for representing the mean velocity distribution $u(y)$ within the main body of the flow. By main body, we mean the intermediate interval of distances y from the wall that are large enough in comparison with viscous layer thicknesses, but small with respect to a characteristic lengthscale of the flow (e.g. tube diameter d). The first form is:

(A) the *scaling* (*power-type*) *law*, depending on the flow Reynolds number,

$$\phi = C\eta^{\alpha}, \tag{1}$$

where

$$\phi = u/u_*, \quad \eta = u_* y/\nu. \tag{2}$$

Here $u_* = (\tau/\rho)^{\frac{1}{2}}$, τ is the shear stress on the wall, ρ is the fluid density, ν is the kinematic viscosity, C and α are dimensionless constants believed to be slowly varying functions of the flow Reynolds number $Re = \bar{u}d/\nu$, and $\bar{u}$ is the mean fluid velocity averaged over the tube (or channel) cross-section. The second form is

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(B) the *universal logarithmic law*

$$\phi = (1/\kappa) \ln \eta + C_1, \tag{3}$$

where κ (the von Kármán constant) and C_1 are believed to be universal constants, i.e. independent of Re.

We emphasize that the derivations of both (A) and (B) are equally rigorous. However, they rest on entirely different basic assumptions. Indeed, both derivations start from similarity and asymptotic considerations, namely the assumption that the mean velocity gradient $\partial_y u$ can depend, in principle, on the following kinematic quantities, only: u_*, y, ν and d. Therefore, from dimensional considerations, one obtains

$$\partial_y u = (u_*/y)\, \Phi(\eta, Re) \quad \text{or} \quad \partial_\eta \phi = (1/\eta)\, \Phi(\eta, Re), \tag{4}$$

where Φ is some dimensionless function of its dimensionless arguments. Here we have replaced the dimensionless parameter $Re_* = u_* d/\nu$, suggested by dimensional analysis, by the flow Reynolds number $Re = \bar{u} d/\nu$, because it follows from dimensional considerations that Re_* is a certain function of Re only.

The further assumptions on which (A) and (B) are based differ, however, in an essential way. The universal law (B) is derived (see e.g. Landau & Lifshiftz 1987; Monin & Yaglom 1971) from the assumption that for sufficiently large local Reynolds number $\eta = u_* y/\nu$ (the observation point y being outside the viscous layer) and sufficiently large flow Reynolds number $Re = \bar{u} d/\nu$ (the turbulent flow being assumed to be fully developed), the dependence of the velocity gradient on the molecular viscosity disappears completely. Various approaches to this derivation, such as the Isaakson and Millikan matching procedures (see Monin & Yaglom 1971), are merely different ways of introducing this very strong assumption, which means that the function $\Phi(\eta, Re)$ tends, as $\eta \to \infty, Re \to \infty$, to a finite limit, different from zero, so that at large η and Re this function can be replaced by its limiting value $\Phi(\infty, \infty) = 1/\kappa$, say, whence the universal logarithmic law (B) follows directly by integration.

According to the alternative assumption used in the derivation of the power law (A) (Barenblatt & Monin 1979; Barenblatt 1979), a finite limit of the function $\Phi(\eta, Re)$ as $\eta \to \infty$, $Re \to \infty$ does not exist. However, at large η (but not so large as to take us out of the intermediate region of the shear flow in which we are interested) and large Re, the function Φ has, according to the alternative assumption, a power-type asymptotic behaviour

$$\Phi \sim A \eta^{\alpha}, \tag{5}$$

where α and A depend somehow on the flow Reynolds number. In fact, we are interested not in the limiting value of Φ, but rather in the intermediate asymptotic behaviour of the function Φ, valid in the intermediate flow region, the main body of the flow referred to previously. If the intermediate asymptotic law (5) is valid, then we substitute (5) into (4) and obtain, by integration, the power law (1). Thus the dependence of the velocity gradient on molecular viscosity, using this assumption, does not disappear, however large both Reynolds numbers, η and Re, may be.

Assumption (5) is an example of an incomplete similarity (or scaling) assumption. Using assumptions of this nature, L. Kadanoff, A. Patashinsky & V. Pokrovsky, and K. Wilson and others were able to obtain remarkable results in quantum field theory and in statistical physics (Amit 1978; Ma 1976). In classical mathematical physics, and in particular in fluid mechanics, the use of such assumptions started from the paper of Guderley (1942) concerning a very intense converging shock wave. In a different context, this idea was introduced even earlier in the papers by Kolmogorov, Petrovsky & Piskunov (1937) and by Fisher (1937) on the propagation of a gene having an advantage in the struggle for life, as well as in the paper of Zeldovich & Frank-Kamenetsky (1938) on the theory of flame propagation.

We re-emphasize that neither the power-law (A) nor the logarithmic law (B) should be considered merely as convenient representations of empirical data. Rather, they have equally rigorous theoretical foundations which are, however, based on essentially different assumptions.

An important question of a qualitative nature arises, therefore, as to whether either of these assumptions is correct.

The results presented in this work give some evidence in favour of the scaling law (1), with an exponent α inversely proportional to the logarithm of flow Reynolds number, Re, and C a linear function of this logarithm. Therefore, the reciprocal of the logarithm of the flow Reynolds number appears naturally as the small parameter of the problem.

2. Analysis

2.1. *The basic conjecture*

The modified theory of the local structure of developed turbulent flows now attracts wide attention, especially considering the corrections to the classical Kolmogorov–Obukhov exponents $\frac{2}{3}$ and $\frac{5}{3}$. An important recent work of Castaing, Gagne & Hopfinger (1990) showed as one of its basic results that these corrections are inversely proportional to $\ln Re_\lambda$, where Re_λ is the Reynolds number based on the Taylor lengthscale λ. (In fact, it is related quantities that have this Re_λ dependence, and not the exponent corrections themselves, but the distinction is immaterial in the present context.) Discussing this work at the recent seminar given by B. Castaing at the Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie in Paris, the present author came to the conclusion that this result, if correct, should be a rather general property of developed turbulent flow, because it is based on general fractal properties of vortex dissipative structures in such flows. In particular, this means that such inverse proportionality to the logarithm of the Reynolds number could be valid for the exponent α in the power law (1). Rough estimates, fitted empirically from experimental data of Nikuradze (1932), given in the book by Schlichting (1968), appear to be in accordance with this conjecture, and will be shown here to be closely consistent with a specific form of the conjecture, in which, namely,

$$\alpha = 3/2 \ln Re. \tag{6}$$

Therefore, without attempting to make the coefficient precise, all sixteen sets of experimental data† available in Nikuradze (1932), of mean velocity measurements at various distances from the wall and at various Reynolds numbers (covering nearly three decimal orders of magnitude), were subjected to a rather severe procedure for the verification of the conjecture (6), namely that the functions $\phi^{[2\ln Re/3]}$ were plotted as functions of η and inspected. The question was whether straight lines in the intermediate interval of η would appear. The processing of experimental data is considered at some length in Part 2 (Barenblatt & Prostokishin 1993). This processing clearly revealed intermediate straight lines for all sixteen sets; see Part 2, figure 1 (*a–e*). The author emphasizes the good accuracy in this: the exponent $1/\alpha = [2 \ln Re]/3$ is large, of the order of ten, or so. Therefore even a small deviation in

† It is essential that the data are presented in the form of tables. The paper by Nikuradze (1932) is unique in this respect.

exponents could destroy the straight lines. By revealing the straight lines in an intermediate interval, we can consider this as direct experimental verification of the basic conjecture (6).

One point should also be explained, and that is that there exists some obvious arbitrariness in the definition of Re. For instance, the maximum velocity can be taken instead of the average or radius instead of diameter. For the relation (6), this arbitrariness is immaterial because (6) should be the first term in the asymptotic expansion, valid when $\ln Re$, and not only Re, is itself large, so that another definition of Re will influence only higher-order terms of the asymptotics. This in itself lends support to the notion that $\ln Re$, rather than some other function of Re, should be involved.

2.2. *The form of the proposed scaling law*

Further processing (see details in Part 2) allows us to obtain, also with rather good accuracy, the following linear dependence of the coefficient C in the power law (1) on $\ln Re$:

$$C = \frac{1}{\sqrt{3}} \ln Re + \frac{5}{2} = \frac{\sqrt{3}+5\alpha}{2\alpha}. \tag{7}$$

Therefore, the power law (1) can be represented in the following form:

$$\phi \equiv \left(\frac{1}{\sqrt{3}} \ln Re + \frac{5}{2}\right) \eta^{3/(2\ln Re)}. \tag{8}$$

Simple transformations reduce (8) to a quasi-universal form involving a new function $\psi(\phi)$:

$$\psi = \frac{1}{\alpha} \ln \frac{2\alpha\phi}{\sqrt{3}+5\alpha} = \ln \eta, \quad \alpha = \frac{3}{2 \ln Re}. \tag{9}$$

This means that if (8)–(9) are true, then all experimental points, corresponding to various but sufficiently large Reynolds numbers η and Re, should settle down on a single curve in the $(\psi, \ln \eta)$-plane (a straight line, in fact), bisectrix of the first quadrant. (We note that plotting $\ln \eta$ (or $\log \eta$) on the abscissa is traditional in representing experimental data for velocity.) As is shown in Part 2 (figure 3), the overwhelming majority of the 256 experimental points available in the tables of Nikuradze (1932) do indeed settle down close to the bisectrix, in accordance with (9). Naturally, the points corresponding to small $\ln \eta$ deviate below the bisectrix, but it should be emphasized that this deviation is systematic.

2.3. *Universal logarithmic law and its relation to the envelope of power-law curves*

In the $(\phi, \ln \eta)$-plane the curves which represent the power laws are different for various Reynolds numbers:

$$\phi = \left(\frac{1}{\sqrt{3}} \ln Re + \frac{5}{2}\right) \exp\left(\frac{3 \ln \eta}{2 \ln Re}\right) = F(\ln \eta, Re). \tag{10}$$

They form a family of curves, which covers a certain part of the $(\phi, \ln \eta)$-plane; the Reynolds number is the parameter of this family. This family possesses an envelope which satisfies both (10) and the equation $\partial_{Re} F = 0$. This last equation can be easily reduced to the form

$$\frac{3 \ln \eta}{2 \ln Re} = \frac{\sqrt{3}}{10} \ln \eta \left[\left(1 + \frac{20}{\sqrt{3} \ln \eta}\right)^{\frac{1}{2}} - 1\right]. \tag{11}$$

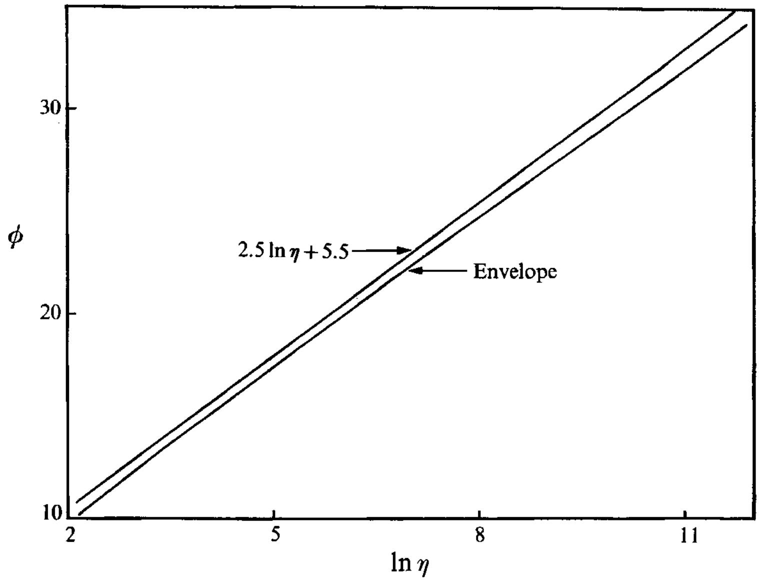

FIGURE 1. Envelope, defined implicitly by letting Re vary in (10) and (11), of the scaling law curves in the (ϕ, ln η)-plane. Each scaling law curve has fixed Re. Although not a straight line, the envelope is very close to the generally accepted universal logarithmic law (the plotted straight line), even at moderate ln η.

The pair of equations (10), (11) parametrizes the envelope. This is plotted as the lower curve in figure 1. As can be seen, even for rather moderate ln η, it is close to the straight line

$$\phi = 2.5 \ln \eta + 5.5, \tag{12}$$

which represents (according to Schlichting 1968) the universal logarithmic law with empirically fitted constants. Even better will be the agreement (the two curves in the figure almost coinciding) if the constant 5.5 in (12) is replaced by the value 5.1, used, for instance, by Monin & Yaglom (1971, p. 273). This agreement seems to be natural from the present perspective because the envelope is the geometric locus of the points at which the derivative with respect to the flow Reynolds number vanishes. However, this was precisely the first of the assumptions on which the derivation of (3) was based!

Moreover, if we allow ln η to tend to infinity while we remain on one of the curves (10) at Re = constant, the function $\Phi(\eta, Re)$, in the general similarity relation (4) tends obviously to infinity. However, if we allow ln η to tend to infinity while we remain on the envelope, ln Re tending simultaneously to infinity, we obtain a finite limit.† Indeed, for large ln η, the relation (11) gives $(3 \ln \eta)/(2 \ln Re) \rightarrow 1$ on the envelope, whence, together with (10), we obtain

$$\phi = \frac{\sqrt{3e}}{2} \ln \eta + \text{const}, \quad \Phi(\eta, Re) \rightarrow \frac{\sqrt{3e}}{2}. \tag{13}$$

Therefore at large ln η, all assumptions leading to the universal logarithmic law are fulfilled on the envelope, and thus this envelope can be identified with the universal logarithmic law. This allows us to obtain the value of the von Kármán constant:

$$\kappa = \frac{2}{\sqrt{3e}} \sim 0.425, \tag{14}$$

† The reader is reminded here that we refer to intermediate asymptotics, and not to limits.

close to what was proposed in Nikuradze (1932), i.e. the experimental value $\kappa = 0.417$. In fact, as we have seen, the envelope is sufficiently close to the straight line (12), even at moderate $\ln \eta$. At very large $\ln \eta$, where $(3 \ln \eta)/(2 \ln Re) \sim 1$, the value of const in (13) is revealed, being equal to $5e/2 \sim 6.79$. However, such values of $\ln \eta$, where $20/\sqrt{3} \ln \eta \ll 1$ and $(3 \ln \eta)/(2 \ln Re) \sim 1$, were never reached in experiments, so the relation (12) (or even more the same straight line with the constant 5.1) is an approximation to the envelope at moderate $\ln \eta$.

2.4. *The skin friction law*

On the basis of the proposed scaling law (8) for the mean velocity distribution, the corresponding skin friction law can be derived and compared with experimental data, in particular with the data available in Nikuradze (1932). The coincidence of experimental data concerning the skin friction with predictions can also contribute to the verification of the proposed scaling law.

We define the dimensionless skin friction coefficient λ in a common way as

$$\lambda = \frac{\tau}{\rho \bar{u}^2/8} = 8\left(\frac{u_*}{\bar{u}}\right)^2. \tag{15}$$

According to (8), the following relation for the mean velocity $\bar{u}$ can be obtained:

$$\begin{aligned}\bar{u} &= \frac{8}{d^2}\int_0^{d/2} u(y)\left(\frac{d}{2}-y\right)dy \\ &= u_* \frac{\sqrt{3}+5\alpha}{\alpha}\left(\frac{u_* d}{\nu}\right)^{\alpha}\frac{1}{2^{\alpha}(1+\alpha)(2+\alpha)}.\end{aligned} \tag{16}$$

In deriving the relation (16), the fact was used, as usual, that for developed turbulent flows in tubes (or channels), the contribution to bulk discharge can be neglected from the viscous layer near the wall, as can the contribution of the tiny region near the tube axis (or symmetry plane in a channel) where the scaling law (8) is not valid.

Because of the basic conjecture (6),

$$Re = \bar{u}d/\nu = \exp(3/2\alpha), \tag{17}$$

whence, together with (16), we obtain

$$\frac{u_* d}{\nu} = \left[\frac{e^{3/2\alpha}\, 2^{\alpha}\, \alpha(1+\alpha)(2+\alpha)}{\sqrt{3}+5\alpha}\right]^{1/(1+\alpha)}, \tag{18}$$

and thus we reach the final relation for the dimensionless skin friction coefficient λ, corresponding to the scaling law (8), as function of the Reynolds numbers:

$$\lambda = 8/\Psi^{2/(1+\alpha)}, \tag{19}$$

where

$$\Psi(\alpha) = \frac{e^{\frac{3}{2}}(\sqrt{3}+5\alpha)}{2^{\alpha}\alpha(1+\alpha)(2+\alpha)}, \quad \alpha = \frac{3}{2 \ln Re}. \tag{20}$$

A comparison between the values of λ predicted by the relation (19) and experimental values λ_e of the skin friction coefficient is presented in Part 2 (figure 5). According to the prediction, the quantity

$$\xi = \lambda_e\, \Psi^{2/(1+\alpha)}/8 \tag{21}$$

should be equal to unity. In fact, the deviations are within the experimental scatter.

3. Conclusions

As we have seen, the scaling (power-type) law (A) has no less rigorous theoretical foundation than the universal logarithmic law. Experimental confirmation of the scaling law proposed here is at least no worse than that of the logarithmic law. However, these laws are based on essentially different physical assumptions: the logarithmic law is based on the assumption that the velocity gradient in the main body of a developed turbulent shear flow is completely independent of the molecular viscosity, and the scaling law is based on the assumption that dependence on molecular viscosity is preserved, although in an asymptotic power-type form. The difference is, as a matter of principle, essential. However, the data presented in Nikuradze (1932) correspond to the range of parameters η, Re, where the difference between the predictions is not too large, because the experimental data are close to the envelope. Meanwhile, the scaling law corresponds to a one-parameter family of curves rather than to the single curve of the universal logarithmic law, so ranges of the parameters η and Re should exist where the difference is significant. Therefore a careful analysis of the available data giving the deviation from the logarithmic law, as well as further specially designed experiments, is needed in order to make a final choice between the two laws and the two assumptions. The extension of the proposed model presented here to heat and mass transfer, rough walls, and the action of polymeric additives, seems to be natural.

The scale law (8) can be represented in the form

$$u = \frac{\sqrt{3}+5\alpha}{2\alpha} u_*^{1+\alpha} \nu^{-\alpha} y^{\alpha}. \tag{22}$$

It is readily seen that we encounter here a situation which typifies incomplete self-similarities (many such examples can be found in Barenblatt 1979), namely that if we reduce molecular viscosity arbitrarily in the complete-similarity case of the universal logarithmic law, the asymptotics for the velocity gradient remain the same: molecular viscosity disappears completely from the list of governing parameters. This is not the case for the scaling law (8). The asymptotics (22) is invariant with respect to a transformation group

$$\nu_1 = \mu\nu, \quad u_1 = \mu^{-\alpha} u, \tag{23}$$

which is precisely what is known as the renormalization group (with μ the group parameter). Taking as small parameter

$$\epsilon = 1/\ln Re, \tag{24}$$

we come to the asymptotic expansions for the coefficients α and C of the scaling law (1), namely

$$\alpha = a_1\epsilon + a_2\epsilon^2 + \ldots, \quad C = c_1/\epsilon + c_2 + c_3\epsilon + \ldots. \tag{25}$$

A regular asymptotic procedure for obtaining the coefficients in (25) would be of special interest. The values of the first coefficients $a_1 = \frac{3}{2}, c_1 = 1/\sqrt{3}, c_2 = \frac{5}{2}$ obtained here should be obtained by this procedure.

As a natural measure of the lengthscale of vortical dissipative structures, the quantity

$$l = (\partial_y u/u_*)^{-1} \tag{26}$$

can be considered. The scaling law (8) gives

$$l = \frac{2}{\sqrt{3}+5\alpha} y^{1-\alpha} \left(\frac{\nu}{u_*}\right)^{\alpha}. \tag{27}$$

Comparing (27) with the relation for the approximation to the length of the fractal curve L_η having fractal dimension $1+\beta$, provided by broken line segments with unit length η,

$$L_\eta = \text{const}\, D^{1+\beta}/\eta^\beta, \tag{28}$$

we can transform (28) into a relation for the diameter D of the fractal curve,

$$D = \text{const}\, L_\eta^{1-\alpha}\eta^\alpha, \quad \alpha = \beta/(1+\beta), \tag{29}$$

similar to (27), so the vortical filament can be considered as a fractal curve with fractal dimension $1/(1-\alpha) = 2\ln Re/(2\ln Re - 3)$. After some transformations we obtain

$$l = \frac{2}{\sqrt{3}+5\alpha}\, y^{1-\alpha} d^\alpha \left[\frac{e^{\frac{3}{2}}(\sqrt{3}+5\alpha)}{2^\alpha\alpha(1+\alpha)(2+\alpha)}\right]^{\alpha/2} e^{-\alpha\ln Re}. \tag{30}$$

For large $\ln Re$, i.e. small α (30) is reduced to a relation

$$l = \frac{2}{\sqrt{3}}\, y\, e^{-\alpha\ln Re} \sim \frac{2}{\sqrt{3}}\, y\, e^{-\frac{3}{2}} = \frac{1}{\kappa\, e^{\frac{1}{2}}}\, y, \tag{31}$$

where the dependence on Re has disappeared.

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